

Indian Institute of Technology Roorkee

Spring Semester 2025-26

MAI-102 (Mathematics II)

Assignment 4

Topics: Concept of probability, random variables and their probability distributions, expectation, moments and moment generating functions, Chebyshev's inequality.

- (1) An experiment has 3 possible mutually exclusive outcomes A, B and C. Check whether the following assignment of probability is correct or not.
- (a) $P(A) = 0.3, P(B) = 0.4, P(C) = 0.34$.
(b) $P(A) = 0.5, P(B) = 0.1, P(C) = 0.5$.
(c) $P(A) = 0.4, P(B) = 0.64, P(C) = -0.04$.
(d) $P(A) = 0.2, P(B) = 0.3, P(C) = 0.5$.
- (2) Examine the consistency of the following probabilities
 $P(A) = 0.4, P(B) = 0.3, P(C) = 0.7, P(A \cap B) = 0.2 = P(B \cap C), P(A \cap C) = 0.4$, and
 $P(A \cap B \cap C) = 0.5$.
- (3) If $P(A) = 0.3, P(B) = 0.5, P(A \cap B) = 0.24$, find $P(A \cup B), P(\bar{A} \cap B), P(\bar{A} \cap \bar{B}), P(\bar{A})$.
- (4) Out of 60 students, 30 are studying Mathematics, while 20 are studying Statistics, and 10 are studying both. If a student is chosen at random, what is the probability that he is studying (i) either Mathematics or Statistics (ii) both Mathematics and Statistics (iii) none of these.
- (5) Two students A and B work independently on a problem. The probability that A will solve is $3/4$, while that of B is $2/3$. If both try to solve the problem, what is the probability that problem will be solved?
- (6) A coin is known to come up heads three times as often as tails. This coin is tossed three times. Let X be the number of heads that appear. Write the probability mass function (pmf) and the cumulative distribution function (cdf) of X .
- (7) Consider an urn which contains slips of paper, each with a number $1, 2, \dots, 100$ on it. Suppose there are i slips with the number i on them for $i = 1, 2, \dots, 100$. The slips are identical except for the numbers. Suppose one slip is drawn at random and let X be the number on the slip. Then
- (a) Find the cdf of X . (b) Compute $P(25 < X \leq 50)$.
- (8) Let the sample space be $S = \{a_1, a_2, a_3, a_4\}$ with $P(a_1) = 1/6, P(a_2) = 1/4, P(a_3) = 1/3$ and $P(a_4) = 1/4$. If X is a random variable defined on S as $X(a_1) = -2, X(a_2) = 0, X(a_3) = 2, X(a_4) = 5$. Find $P(-3 \leq X \leq 3), P(0 \leq X \leq 4), P(2 \leq X \leq 10)$.
- (9) Obtain the probability mass function of a random variable X which takes the values -3, -2, -1, 0, 1, 2 and 3 such that $P(x = 0) = P(x < 0) = P(x > 0)$; $P(X = -3) = P(x = -2) = P(X = -1)$ and $P(X = 3) = P(x = 2) = P(X = 1)$.
- (10) Let the probability density function (pdf) of a continuous random variable X be

$$f(x) = \begin{cases} \alpha x, & 0 \leq x < 1, \\ \alpha, & 1 \leq x < 2, \\ 3\alpha - \alpha x, & 2 \leq x \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Determine α , and the cdf of X .
 (b) If x_1, x_2 and x_3 are 3 independent observations from X , what is the probability that exactly one of these 3 observations is greater than 1.5?

(11) Suppose a continuous random variable X has the pdf

$$f(x) = \begin{cases} kx^2e^{-\frac{x}{2}}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where k is a suitable constant. (a) Determine the cdf of X , (b) Compute the probability that X is at most 10 given that X is at least 5.

(12) (a) Two distinct integers are chosen at random and without replacement from the first five positive integers. Let X be the absolute value of the difference of these two numbers. Compute the mean and the variance of X .

(b) Let a random variable X has the pdf $f(x) = \frac{3}{8}x^2, 0 < x < 2$, zero elsewhere. Consider a random rectangle whose sides are X and $2 - X$. Determine the expected value of the area of the rectangle.

(13) Five devices are subjected to successive reliability tests. Each device is tested only if the preceding one passes the reliability test. Determine the expected number of reliability tests done if the probability of passing the reliability test is 0.9 for each device.

(14) Find the value of $k, E(x), V(x)$, cumulative distribution function of the random variable X having probability density function as

$$f(x) = \begin{cases} kx, & 0 \leq x \leq 2, \\ k(4 - x), & 2 \leq x \leq 4, \\ 0, & \text{elsewhere} \end{cases}$$

(15) The moments of a random variable X are given by

$$E[X^k] = p, \quad k = 1, 2, 3, \dots, \text{ where } 0 < p < 1.$$

(a) Find the moment generating function (mgf) of X .

(b) Identify the probability mass function (pmf) and cumulative distribution (cdf) of X .

(16) Let X be a random variable with the pdf $f(x) = \frac{1}{2}e^{-|x|}, -\infty < x < \infty$. Determine the moment generating function (mgf) of X .

(17) (a) The moment generating function (mgf) of random variable X is $M_X(t) = (1 - t)^{-3}, |t| < 1$. Find the first four moments of X about origin. Also calculate the mean and variance of X .

(b) The mgf of a discrete random variable X is $\frac{2e^t}{3 - e^t}, t < \log 3$. Identify the pmf of X .

(c) Given that the mgf of a random variable X is $M_X(t) = e^{3t + 8t^2}$. Find the mgf of the random variable $Z = \frac{1}{4}(X - 3)$. Use it to calculate the mean and variance of Z .

(18) (a) If X is a random variable such that $E(X) = 3$ and $E(X^2) = 13$, use Chebyshev's inequality to find (i) a lower bound for $P[-2 < X < 8]$ (ii) an upper bound for $P[|X - 3| \geq 7]$.

(b) Does there exist a random variable X with mean 5 and variance 9 such that $P[-1 < X < 11] = 0.6$? Explain.

(19) Let X be a discrete random variable with pmf $P[X = -1] = \frac{1}{8}, P[X = 0] = \frac{3}{4}, P[X = 1] = \frac{1}{8}$, zero elsewhere. Verify that Chebyshev's inequality gives an exact bound for $P[|X - \mu| \geq 2\sigma]$.

ANSWERS

(1) (a) no (b) no (c) no (d) yes

(2) Not consistent.

(3) $P(A \cup B) = 0.56$, $P(\bar{A} \cap B) = 0.26$, $P(\bar{A} \cap \bar{B}) = 0.44$, $P(\bar{A}) = 0.7$.(4) (i) $\frac{2}{3}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{3}$ (5) $\frac{11}{12}$

(6) Pmf:

x	:	0	1	2	3
$f(x)$:	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{27}{64}$	$\frac{27}{64}$

Cdf:

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1/64, & \text{for } 0 \leq x < 1 \\ 10/64, & \text{for } 1 \leq x < 2 \\ 37/64, & \text{for } 2 \leq x < 3 \\ 1, & \text{for } x \geq 3. \end{cases}$$

(7) (a) Cdf:

$$F(x) = \begin{cases} 0, & \text{for } x < 1, \\ \frac{|x|(|x|+1)}{10100}, & \text{for } 1 \leq x < 100, \\ 1, & \text{for } x \geq 100. \end{cases}$$

OR

$$F(x) = \begin{cases} 0, & \text{for } x < 1, \\ \frac{i(i+1)}{10100}, & \text{for } i \leq x < i+1, \text{ for } i = 1, 2, \dots, 100 \\ 1, & \text{for } x \geq 100. \end{cases}$$

(b) $\frac{19}{101}$ (8) $\frac{3}{4}$ $\frac{7}{12}$ $\frac{7}{12}$

(9) Pmf:

x	:	-3	-2	-1	0	1	2	3
$f(x)$:	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

(10) (a) $\alpha = \frac{1}{2}$

Cdf:

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ x^2/4, & \text{for } 0 \leq x < 1 \\ (2x-1)/4, & \text{for } 1 \leq x < 2 \\ (6x-x^2-5)/4, & \text{for } 2 \leq x < 3 \\ 1, & \text{for } x \geq 3. \end{cases}$$

(b) $\frac{3}{8}$

(11) (a) Cdf:

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1 - \left(\frac{x^2+4x+8}{8}\right) e^{-\frac{x}{2}}, & \text{for } x \geq 0 \end{cases}$$

(b) $1 - \frac{148}{53} e^{-\frac{5}{2}}$ (12) (a) Mean $X = 2$, $\text{Var}(X) = 1$ (b) $\frac{3}{5}$

(13) 4.0951

$$(14) \quad k = \frac{1}{4}, \quad E(X) = 2, \quad V(X) = \frac{2}{3}.$$

Cdf:

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{x^2}{8}, & \text{for } 0 \leq x < 2 \\ x - \frac{x^2}{8} - 1, & \text{for } 2 \leq x < 4 \\ 1, & \text{for } x \geq 4. \end{cases}$$

$$(15) \quad (a) \quad \text{MGFM}_X(t) = 1 - p + pe^t, \quad \forall t;$$

$$(b) \quad p_X(0) = 1 - p, \quad p_X(1) = p.$$

Cdf:

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1 - p, & \text{for } 0 \leq x < 1 \\ 1, & \text{for } 1 \leq x < \infty. \end{cases}$$

$$(16) \quad M_X(t) = \frac{1}{1-t^2}, \quad |t| < 1$$

$$(17) \quad (a) \quad \mu'_1 = 3, \quad \mu'_2 = 12, \quad \mu'_3 = 60, \quad \mu'_4 = 360, \quad \text{Mean } X = 3, \quad \text{Var}(X) = 3$$

$$(b) \quad p(x) = 2 \left(\frac{1}{3}\right)^x, \quad x = 1, 2, 3, \dots$$

$$(c) \quad M_Z(t) = e^{\frac{t^2}{2}}, \quad \text{Mean } Z = 0, \quad \text{Var}(Z) = 1$$

$$(18) \quad (a) \quad (i) \frac{21}{25}, \quad (ii) \frac{4}{49}$$

(b) No