Indian Institute of Technology Roorkee
Spring Semester 2025-26
MAI-102 (Mathematics II)
Assignment 3

(*Topics: Inner-product spaces, Gram-Schmidt process, orthonormal basis; spectral theorem for real symmetric matrices*)

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(1) Provide reasons why each of the following is not an inner product on the given vector spaces.
 (a) $\langle (a,b),(c,d)\rangle = ac - bd$ on $\mathbb{R}^2$.
 (b) $\langle A, B\rangle = \mathrm{Tr}(A+B)$ on $M_{2\times 2}(\mathbb{R})$.
 (c) $\langle f, g\rangle = \int_0^1 f'(x)g(x)dx$ on $\mathcal{P}(\mathbb{R})$ where $'$ denotes differentiation.
 (d) $\langle f, g\rangle = \int_0^{1/2} f(x)g(x)dx$ on $C([0,1])$ over $\mathbb{R}$.

(2) (a) Let $V = M_n(\mathbb{C})$ be the set of all $n\times n$ matrices with entries from $\mathbb{C}$ and for $A, B \in M_n(\mathbb{C})$, define $\langle A, B\rangle = \mathrm{Tr}(B^*A)$, where B^* denotes the conjugated transpose of B. Verify that $\langle\cdot,\cdot\rangle$ defines an inner product (Frobenius inner product) in V.
 (b) Use the Frobenius inner product to compute $\|A\|$, $\|B\|$ and $\langle A, B\rangle$ for

$$A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix},\ B = \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}.$$

 (c) In $\mathbb{C}^2$, show that $\langle x, y\rangle = xAy^*$ is an inner product where $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$. Compute $\langle x, y\rangle$ for $x = (1-i, 2+3i)$ and $y = (2+i, 2-3i)$.

(3) In each of the following, use Gram-Schmidt process to convert the given basis into an orthonormal basis, using standard inner product.
 (a) $\{(1,0),\ (1,-1)\}$, in $\mathbb{R}^2$;
 (b) $\{(1,1,1),(0,1,1),(0,0,1)\}$ in $\mathbb{R}^3$;
 (c) $\{(1,1,0,1),\ (1,-2,0,0),\ (1,0,-1,2),\ (0,0,0,1)\}$ in $\mathbb{R}^4$.

(4) (a) (i) In $\mathbb{R}^2$, let $\alpha = (a_1, a_2),\ \beta = (b_1, b_2)$. Determine whether $\langle\cdot,\cdot\rangle$ is a real inner product for $\mathbb{R}^2$ if $\langle\cdot,\cdot\rangle$ be defined by

$$\langle \alpha, \beta\rangle = a_1b_1 + 2a_1b_2 + 2a_2b_1 + 5a_2b_2.$$

 (ii) Use Gram-Schmidt process to obtain an orthonormal basis from the basis set $\{(1,0),\ (1,-1)\}$ of $\mathbb{R}^2$ with the above inner product.

 (b) (i) In $\mathbb{R}^3$, let $\alpha = (a_1, a_2, a_3),\ \beta = (b_1, b_2, b_3)$. Determine whether $\langle\cdot,\cdot\rangle$ is a real inner product for $\mathbb{R}^3$ if $\langle\cdot,\cdot\rangle$ be defined by

$$\langle \alpha, \beta\rangle = a_1b_1 + (a_2+a_3)(b_2+b_3) + a_3b_3.$$

 (ii) Use Gram-Schmidt process to obtain an orthonormal basis from the basis set $\{(1,0,1),\ (1,1,1),\ (2,6,8)\}$ of $\mathbb{R}^3$ with the above inner product.

(5) Let $\mathcal{P}_2[x]$ denote the set of polynomials in x, with real coefficients, of degree at most 2.
 (a) Check that $\langle\cdot,\cdot\rangle$ defined by $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$ is an inner product on $\mathcal{P}_2[x]$.
 With respect to this inner product above, use Gram-Schmidt process to obtain an orthonormal basis from the basis set $\{1, x, x^2\}$.
 (b) Check whether $\langle\cdot,\cdot\rangle$ defined by $\langle f, g\rangle = \int_0^1 \frac{f(t)g(t)}{\sqrt{1-t^2}}dt$, is also an inner product on $\mathcal{P}_2[x]$.

(6) Consider the subset $S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}$ of $M_2(\mathbb{R})$, where $M_2(\mathbb{R})$ denotes the set of all 2×2 matrices with real entries.

With respect to the inner product $\langle A, B \rangle = \mathrm{Tr}(B^\top A)$ for $A, B \in M_2(\mathbb{R})$, obtain an orthonormal basis for $\mathrm{span}(S)$, using Gram- Schmidt process.

(7) Prove that in an inner product space V, an orthogonal set S is a linearly independent set.

(8) Prove that for all $\alpha,\ \beta$ in a real vector space V,
(a) $\langle \alpha, \beta \rangle = 0$ if and only if $\|\alpha + \beta\| = \|\alpha - \beta\|$,
(b) $\langle \alpha + \beta,\ \alpha - \beta \rangle = 0$ if and only if $\|\alpha\| = \|\beta\|$.

(9) (a) For an inner product space over V (over $\mathbb{R}$ or $\mathbb{C}$), prove the following:
(i) $|\langle x, y \rangle| \leq \|x\|\|y\|$ for all $x, y \in V$. (Cauchy-Schwarz inequality)
(ii) $|\langle x, y \rangle| = \|x\|\|y\|$ if and only if one of the vectors x or y is a multiple of the other.
(iii) $\|x + y\| \leq \|x\| + \|y\|$. (triangle inequality)
(b) Let $C([0, 1])$ denote the space of all real-valued continuous functions defined on the closed interval $[0, 1]$. Show that $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$ defines a norm on $C([0, 1])$. Let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g \rangle$, $\|f\|$, $\|g\|$ and $\|f + g\|$. Then verify both the Cauchy-Schwarz inequality and the triangle inequality.

(10) Let V be an inner product space.
(a) Prove that $\|x \pm y\|^2 = \|x\|^2 \pm 2\mathrm{Re}(\langle x, y \rangle) + \|y\|^2$, for all $x, y \in V$, where $\mathrm{Re}(\langle x, y \rangle)$ denotes the real part of the complex number $\langle x, y \rangle$.
(b) Suppose that x and y are orthogonal vectors in V. Prove that

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Deduce Pythagorean theorem in $\mathbb{R}^2$.
(c) Prove the parallelogram law on an inner product space V; that is, show that

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad \text{for all} \quad x, y \in V.$$

What does this equation state about parallelograms in $\mathbb{R}^2$?
(d) If V is an inner product space over $\mathbb{R}$, prove the following polar identity:

$$\langle x, y \rangle = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2.$$

(e) If V is a complex inner product space, the show that

$$\langle x, y \rangle = \frac{1}{4}\left[\|x + y\|^2 - \|x - y\|^2 + \|x + iy\|^2 i - \|x - iy\|^2 i\right].$$

(11) Suppose V is an inner product space and $u, v \in V$ are such that $\|u\| = 2$, $\|u + v\| = 6$, $\|u - v\| = 4$, then what is the value of $\|v\|$?

(12) Verify spectral theorem for the following real symmetric matrices.
(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
(b) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix}$

ANSWERS

(2): (b) $4, 2, -4i$ **(c)** $-6 + 19i$

(3): (a) $\{(1,0),\ (0,-1)\}$,

(b) $\left\{\frac{\sqrt{3}}{3}(1,1,1), \frac{\sqrt{6}}{6}(-2,1,1), \frac{\sqrt{2}}{2}(0,-1,1)\right\}$.

(c) $\left\{\frac{1}{\sqrt{2}}(1,0,1,0), \frac{1}{\sqrt{2}}(0,1,0,1), \frac{1}{\sqrt{2}}(-1,0,1,0)\right\}$.

(4): (a) $\{(1,0),\ (2,-1)\}$

(b) $\{\frac{1}{\sqrt{3}}(1,0,1),\ \frac{1}{\sqrt{6}}(-1,3,-1),\ \frac{1}{\sqrt{2}}(-1,-1,1)\}$

(5): (a) $\left\{1, 2\sqrt{3}(x-\frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6})\right\}$.

(6): $\left\{\frac{1}{6}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \frac{1}{6\sqrt{2}}\begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}, \frac{1}{9\sqrt{2}}\begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}\right\}$

(11): $\sqrt{22}$