Indian Institute of Technology Roorkee
Spring Semester 2025-26
MAI-102 (Mathematics II)
Assignment 2

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(1) Which of the following maps $F : \mathbb{R}^3 \to \mathbb{R}^2$ are linear transformations? Justify your answer.
  (a) $F(x, y, z) = (y, z)$
  (b) $F(x, y, z) = (x + y, -z)$
  (c) $F(x, y, z) = (x + 3, z + 2y)$
  (d) $F(x, y, z) = (|x|, y + z)$
  (e) $F(x, y, z) = (2x + y, 3z)$
  (f) $F(x, y, z) = (\cos x, 0, 0)$
  (g) $F(x, y, x) = (\sin x, \sin y)$

(2) (a) If $F : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $F(x, y, z) = (yz, x^2)$, then find $F(-1, 2, 3)$ and $F^{-1}(0, 0)$. Is F a linear transformation?
(b) Suppose there is a one-one linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$. Find $F^{-1}(0, 0)$.

(3) (a) Give an example of linear transformation T on $\mathbb{R}^3$ such that $T^2 \neq 0$ but $T^3 = 0$.
  (b) Give examples of transformations S and T on $\mathbb{R}^2$ such that $ST \neq 0$ and $TS = 0$.

(4) Let V be the vector space of all $n \times n$ complex matrices. Let B be a fixed matrix in V. Define $T(A) = AB - BA$. Show that T is linear.

(5) Does there exist:
  (a) an injective linear map $T : \mathbb{C}^2 \to \mathbb{C}^3$?
  (b) an injective linear map $T : \mathbb{C}^3 \to \mathbb{C}^2$?
  (c) a surjective linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$?
  (d) a surjective linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$?

(6) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (x + 2y - z, y + z, x + y - 2z).$$

Find a basis and the dimension of
  (a) $R(T)$, range/image of T,
  (b) $N(T)$, kernel/null space of T.

(7) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by
$T(x, y, z, t) = (x - y + z + t, x - y + z + t, 2x - 2y + 3z + 4t, 3x - 3y + 4z + 5t)$.
Find a basis of
  (a) $R(T)$, range/image of T,
  (b) $N(T)$, kernel/null space of T.

(8) Write matrices of the linear maps below. Find inverses, where they exist:
  (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x - y, x - 2y)$
  (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, x - 2y, 3x + y)$
  (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + y - 2z, x + 2y + z, 2x + 2y - 3z)$

(9) Consider the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (3x + 4y, 2x - 5y)$.
Let $B_1 = \{(1, 0), (0, 1)\}$ and $B_2 = \{(1, 2), (2, 3)\}$.
  (a) Find the matrix A of T with respect to basis B_1.
  (b) Find the matrix B of T with respect to basis B_2.

(10) Consider the vector space $P_3[x]$ of all polynomials with real coefficients and of degree atmost 3. Let D denote the differential operator on $P_3[x]$. If $B_1 = \{1, x, x^2, x^3\}$ and $B_2 = \{1 + x, x + x^2, x^2 + x^3, x + x^3\}$, then
  (a) Verify that B_1 and B_2 are bases of $P_3[x]$.
  (b) Find the matrix A_1 of D relative to B_1.
  (c) Find the matrix A_2 of D relative to B_2.

(11) If a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ is such that

$$\ker(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = 5x_2, x_3 = 7x_4\},$$

then prove that T is surjective.
Does there exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^2$ such that

$$\ker(T) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = 3x_2, x_3 = x_4 = x_5\}?$$

(12) (a) Verify Rank-Nullity theorem for $T : P_2[x] \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}.$$

(b) Let $T : P_2[x] \to P_3[x]$ be a linear transformation defined by

$$T(f)(x) = 2f'(x) + \int_0^x 3f(t)dt.$$

Show T is one-one but not onto.

(13) If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$, then find $T(8, 11)$.
Does there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $T(1, 0, 3) = (1, 1)$ and $T(-2, 0, -6) = (2, 1)$?

(14) (a) Let $P[x]$ be the vector space of all polynomials with real coefficients. Define $T : P[x] \to P[x]$ by

$$Tf(x) = \int_0^x f(t)dt.$$

Check whether T is linear transformation. Is T one-one? Is T onto?
(b) Let $T : P[x] \to P[x]$ be defined by $Tf(x) = f'(x)$. Show that T is linear, onto but not one-to-one.

Answers

(1): (a), (b)
(2): (a) z-axis and y-axis, no (b) $\{0\}$
(3): (a) T(x, y, z) = (y+z, z, 0) (not unique) (b) find two such matrices of order 2.
(5): (a) yes (b) no (c) no (d) yes
(6): (a) (1,0,1),(0,1,-1) (not unique), dim=2 (b) (3,-1,1), dim=1
(7): (a) (1,1,1),(0,1,2) (not unique), dim=2 (b) (2,1,-1,0),(1,2,0,1) (not unique), dim=2
(8): (a) $T^{-1}(x, y) = (2x - y, x - y)$ (b) does not exist (c) $T^{-1}(x, y, z) = (-8x - y + 5z, 5x + y - 3z, -2x + z)$
(9): (a) $A = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}$ (b) $B = \begin{pmatrix} -49 & -76 \\ 30 & 47 \end{pmatrix}$
(10) (b) $A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (c) $A_2 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 2 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -2 \end{pmatrix}$
(13): (a) T(8,11)=(5,-3,16) (b) No