Indian Institute of Technology Roorkee

Spring Semester 2025–26

MAI–102 (Mathematics II)

Assignment–1

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(1) Determine whether the following sets are vector spaces over $\mathbb{R}$ with respect to the given operations:

(a) The set $\{(a,b) \in \mathbb{R}^2 : b = 5a + 1\}$ with usual addition and scalar multiplication.

(b) $M_n(\mathbb{R})$, the set of all $n \times n$ real matrices over $\mathbb{R}$ with usual operations.

(c) The set of real sequences $\{u_k\}$ that satisfy the recurrence relation

$$u_{k+1} = u_k + u_{k-1} \qquad \text{for } k \geq 1$$

(2) Determine whether the following sets are subspaces of $\mathbb{R}^4$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^4$.

(a) $W_1 = \{(a,b,c,d) : a + b = c + d\}$

(b) $W_2 = \{(a,b,c,d) : a + b = 1\}$

(c) $W_3 = \{(a,b,c,d) : a^2 = b^2\}$

(3) (a) Which of the following sets of vectors in $\mathbb{R}^3$ are linearly independent ?

i. $\{(1,3,0),(2,-3,4),(3,0,4)\}$,

ii. $\{(1,2,3),(2,3,1),(3,1,2)\}$,

(b) Let $\mathbb{V} := \mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$. Which of the following sets are linearly independent in $\mathbb{V}$?

i. $\{f_1, f_2, f_3\}$, where $f_1(x) = 5x^2 + x + 1, f_2(x) = 2x + 3$ and $f_3(x) = x^2 - 1$.

ii. $\{g_1, g_2, g_3\}$, where $g_1(x) = \cos^2(x), g_2(x) = \cos(2x)$ and $g_3(x) = 1$.

(4) Describe the smallest subspace of $M_2(\mathbb{R})$ containing

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(5) Show that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, $\mathbb{R}^3$, all lines through the origin, and all planes through the origin.

(6) Let U_1, U_2, W be subspaces of V.

(a) Prove or disprove: if $V = U_1 \oplus W = U_2 \oplus W$, then $U_1 = U_2$.

(b) Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$, where U_e and U_o denotes the set of even and odd functions on $\mathbb{R}$ respectively.

(7) Verify that

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

span $M_2(\mathbb{R})$.

(8) Show that the span of

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the space of all symmetric 2×2 matrices.

(9) Find the subspace of P_2 spanned by $p_1(x) = 1 + 3x$, $p_2(x) = x + x^2$, and decide whether $\{p_1, p_2\}$ is a spanning set for P_2.

(10) Show that the solution set of

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_1 - 3x_2 + x_3 = 0 \end{cases}$$

is a subspace of $\mathbb{R}^3$ and find a basis.

(11) Find a basis for the span:

(a) $\{(1, 3, 3), (-3, -9, -9), (1, 5, -1), (2, 7, 4), (1, 4, 1)\}$
(b) $\{(1, 1, -1, 2), (2, 1, 3, -4), (1, 2, -6, 10)\}$

(12) Let

$$p_1(x) = x^3 - x, \quad p_2(x) = x^3 + x^2, \quad p_3(x) = 2x^3 - x^2 + x$$

be polynomials in P_3.

(a) Show that the set $\{p_1, p_2, p_3\}$ is linearly independent in P_3.
(b) Find a polynomial $p_4 \in P_3$ such that $p_4(0) = p_4(1) = 0$ and

$$\{p_1, p_2, p_3, p_4\}$$

forms a basis of P_3.

(13) (a) Find a basis and dimension of the subspace

$$W = \{(a + b + 2c, 2a + 2b + 4c + d, b + c + d, 3a + 3c + d) : a, b, c, d \in \mathbb{R}\}.$$

(b) Let V be a vector space and let $\{u, v, w\}$ be a basis of V. Prove that the sets

$$\{\, u + v,\ u - v,\ u - 2v + w \,\} \quad \text{and} \quad \{\, u + v - 3w,\ u + 3v - w,\ v + w \,\}$$

are also bases of V.

(c) Find a basis of the space $U = \{p \in P_4(\mathbb{R}) : p(6) = 0\}$.

(14) Show that the space $C[0, 1]$, consisting of all real-valued continuous functions defined on the closed interval $[0, 1]$, is infinite-dimensional as a vector space over $\mathbb{R}$.

Answers

(1) (a) No (b) Yes (c) Yes

(2) (a) Yes (b) No (c) No

(3) (a) i. No ii. Yes
(b) i. Yes ii. No

(4) $\left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$

(5) –

(6) (a) Counterexample: $V = \mathbb{R}^2, W = \{(x,0) : x \in \mathbb{R}\}, U_1 = \{(0,y) : y \in \mathbb{R}\}, U_2 = \{(x,y) \in \mathbb{R}^2 : x = y\}$.

(7) Yes

(8) –

(9) Not a spanning set

(10) $\{(1,1,1)\}$

(11) (a) $\{(1,3,3),(1,5,-1)\}$ (b) $\{(1,1,-1,2),(2,1,3,-4)\}$

(12) (b) $x^2 - x$

(13) (a) $\{(1,2,0,3),(1,2,1,0),(0,1,1,1)\}$ and dimension is 3 (b) – (c) $\{(x-6), x(x-6), x^2(x-6), x^3(x-6)\}$

(14) –