A Mathematical Model for Newtonian and Non-Newtonian Flow through Tapered Tubes

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Abstract

An investigation about the blood flow in tapered tubes is modeled in this work. This model represents a steady and laminar flow of Newtonian and non-Newtonian fluid, Navier stokes equations are used in the formulation part of the model. In this work, we take Re numbers up to 10 for Newtonian fluid and 5.7 for non-Newtonian fluid and taper angle \( \alpha \) is 0.5, 0.75, 1.0, 1.25. Pressure gradient, pressure drop and flux increases with distance into the taper as the radius decreases in this model. Pressure drop flow rate data compared with the predictions of a semi-empirical flow model. The semi-empirical model provided much better estimates of pressure drop, pressure gradient and flux. Here Newtonian and non-Newtonian fluids were examined. The experimental data were compared with theoretical predictions. Keywords: Newtonian and non-Newtonian fluid, tapered tubes

1. Introduction

The study of flow through tapered conduits is important not only for an understanding of the blood flow behavior in arteries but also for the design of prosthetic blood vessels. Localized narrowing in a blood vessel is commonly referred to as a stenosis. This vascular disease is frequently occur, particular in mammalian arteries. The intimal thickening of stenotic artery was understood as an early process in the beginning of atherosclerosis.

In this study, pressure drop, pressure gradient and flux were measured in rigid walled model of tapered grafts under steady flow conditions. Both Newtonian and non-Newtonian fluids were examined and the experimental data were compared with theoretical predictions pressure drop flow rate data were obtained in all four test sections for both Newtonian and non-Newtonian flow. Walawender et. al. (1972) worked on experimental studies on the blood flow throughout tapered tubes. Oka (1973) gave an idea pressure development in a non-Newtonian flow through a tapered tube, while Walawender and Chen (1975) discussed about blood flow in tapered tubes. Wang et. al. (2004) worked on numerical study of pulsating flow through a tapered artery with stenosis. Mandal (2005) discussed about an unsteady analysis of non-Newtonian blood flow through tapered arteries with a stenosis. Kumar et al. (2006) worked on the numerical study of the axisymmetric blood flow in a constricted rigid tube, while Sankar and Hemalatha (2007) investigated a non-linear mathematical model for blood flow through tapered tubes.

2. Mathematical model

In this model tapered tubes cast in square section moulds and take taper angles were
0.5, 0.75, 1.0, 1.25 and these valves are called nominal. The tube dimensions are the actual \( \alpha \) are given in table and the geometry of the tapered artery in figure (1).

![Figure (1): The geometry of the tapered artery](image)

<table>
<thead>
<tr>
<th>Normal (( \alpha ))</th>
<th>Actual (( \alpha ))</th>
<th>( R_1 ) (mm)</th>
<th>( L_1 ) (cm)</th>
<th>( L_2 ) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>0.508</td>
<td>4.72</td>
<td>53.18</td>
<td>37.22</td>
</tr>
<tr>
<td>0.75</td>
<td>0.727</td>
<td>4.60</td>
<td>36.25</td>
<td>20.29</td>
</tr>
<tr>
<td>1.0</td>
<td>0.972</td>
<td>4.43</td>
<td>26.11</td>
<td>12.26</td>
</tr>
<tr>
<td>1.25</td>
<td>1.215</td>
<td>4.37</td>
<td>20.61</td>
<td>10.61</td>
</tr>
</tbody>
</table>

Here we also assume that the fluid is incompressible and Newtonian. The flow is fully developed laminar axially symmetric flow, we will employ circular cylindrical co-ordinate \( r, \theta, z \) be state the problem mathematically and choose the \( Z \)-axis along the axis of the tube. The flow of the fluid is governed by the Navier-stokes equations:

\[
\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + V_z \frac{\partial V_r}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left( \frac{\partial^2 V_r}{\partial r^2} + \frac{\partial^2 V_r}{\partial z^2} + \frac{1}{r} \frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} \right) \tag{2.1}
\]

\[
\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left( \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r} - \frac{\partial^2 V_z}{\partial z^2} \right) \tag{2.2}
\]

The equation of continuity is:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r V_r \right) + \frac{\partial V_z}{\partial z} = 0 \tag{2.3}
\]

Here \( V_r \) and \( V_z \) is the radial and axial velocities, \( P \) is the pressure, \( \rho \) is the density of the fluid, \( \nu = \frac{\mu}{\rho} \) is the Kinematic viscosity of the fluid. Oka and Murata (1969) presented general theory of steady non-Newtonian flow through tapered tubes but their solution was restricted to small angles of taper \((\alpha^2 << 1)\), they observed an expression for \( Q \) the flow rate through any cross section.

The general formula of volumetric flow through a slightly tapering tube for any time independent fluid characterized by \( F(t) \):
Here \( \tau_w \) is the wall shear stress which is function of \( z \) and \( \tau \) is the shear stress and \( R \) is the radius of the tapered tube at a point \( z \).

Then \( \tau_w \) is given as:

\[
\tau_w = -\frac{1}{2} R(z) \left( \frac{dp}{dz} \right)
\]

(2.5)

For Newtonian fluid \( \tau \) is given by:

\[
\tau = \mu \frac{\partial V}{\partial r}
\]

(2.6)

Putting this value in equation (2.4) and integrate then we obtained rate of volumetric flow:

\[
Q = \frac{\pi R^3(z)}{8 \mu} \left( \frac{dp}{dz} \right)
\]

(2.7)

Figure (2): Plot of volumetric flow rate against radius of tapered artery for Newtonian flow

The pressure drop \( \Delta P \) is given as:

\[
\Delta P = \frac{8 \mu Q}{\pi} \int_0^z R^{-4}(z) \, dz
\]

(2.8)

According to Figure (1) \( R(z) = R_0 - z \tan \alpha \) and for small values of \( \alpha \),

\[
R(z) = R_0 - \alpha z, \quad \text{where} \quad R_0 = \alpha L_1
\]

then we get:

\[
\Delta P = \frac{8 \mu Q}{3 \mu \alpha^3} \left( L_2^{-3} - L_1^{-3} \right)
\]

(2.9)

For non-Newtonian fluid which obeys the power law equation:

\[
F(t) = \frac{du}{dr} = -\kappa \tau^3
\]

(2.10)

Putting this value in equation (2.4), then we get rate of volumetric flow:
Figure (3): Plot of volumetric flow rate against radius of tapered artery for Non-Newtonian flow

Simplified this equation and integrate then we obtain:

\[ \Delta P = \frac{2n}{3a^{(n+3)/n}} \left[ \frac{Q(n+3)}{\pi R^3} \right]^{(n+3)/n} \left[ L_{i_1} - \frac{L_{i_2}}{z} \right] \] (2.12)

3. Pressure gradient

In a tapered tube pressure gradient is increases with distance into the taper. In a uniform cylindrical tube, the pressure gradient is constant for a power law fluid. The pressure gradient is given as:

\[ \left( \frac{dP}{dz} \right)_{i \to 0} = 1 + \frac{n+3}{n} \frac{a z}{R_0} \]

If we take Newtonian fluid, then \( n = 1 \), the pressure gradient is:

\[ \left( \frac{dP}{dz} \right)_{i \to 0} = 1 + 4 \left( \frac{a z}{R_0} \right) \]

The graph clearly shows the effect of pressure gradient for Newtonian and non-Newtonian fluid.

Figure (4): Plot of pressure drop against Reynolds number for Newtonian flow
4. Result

In this study, we obtain pressure drop, flux and pressure gradient with rigid tapered grafts (for Newtonian and non-Newtonian fluids). The results show that if we take Newtonian fluid flux, pressure drop and pressure gradient are increases. These increments are due to small, if we take non-Newtonian fluid with tapered grafts then the taper of tubes does not change the flow pattern but only changes the value.

References