Forced flow of a Visco-Elastic Second-Grade Fluid between two Infinite Discs

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Abstract

Numerical solution for the steady forced flow of a visco-elastic second-grade fluid between two infinite discs has been obtained by using finite difference method. Variation of the velocity components for different values of the forced parameter, Reynolds number and visco-elastic parameter have been obtained and shown graphically.

Key words: Second-grade fluid, finite difference method, forced flow, infinite disc.

1. Introduction

The constitutive equation for an incompressible second-order fluid is given by

\[ \tau = -pI + \mu_1 A_1 + \mu_2 A_2 + \mu_3 A_1^2, \]

where \(-p\) is a hydrostatic pressure, \(\tau\) is stress tensor and \(A_1, A_2\) are first two Rivlin-Ericksen tensors defined by,

\[ A_1 = \nabla V + (\nabla V)^\prime, \]
\[ A_2 = \frac{dA_1}{dt} + A_1(\nabla V) + (\nabla V)^\prime A_1, \]

where \(V\) is the velocity, \(\mu_1, \mu_2\) and \(\mu_3\) is the coefficient of Newtonian-viscosity, elastico-viscosity and cross-viscosity respectively.

If the fluid modeled by equation (1) is to be compatible with thermo-dynamics, in the sense that all motions of the fluid meet Clausius-Duhem inequality[1] and the assumption that the specific Helmholtz free energy of the fluid is minimum when the fluid is locally at rest, then the condition

\[ \mu_1 \geq 0, \ \mu_2 \geq 0, \ \mu_2 + \mu_3 = 0 \]

must hold. The fluids satisfying (1), (2) and (3) are termed as second grade fluids. The purpose of the present problem is to investigate the forced flow of a second-grade fluid between two infinite discs and the motion is axially symmetric. The equation of motion for two different cases,

(i) when upper disc rotates and the lower disc is stationary and
(ii) when lower disc rotates and upper disc is stationary,

has been solved by finite difference method. The visco-elastic effects and those of forced flow on the velocity components have been investigated and shown graphically through figs.1 and 2.

2. Formulation of the problem

The equation of motion and continuity are,

\[ \rho \frac{dV}{dt} = \nabla \cdot \tau \]
\[ \nabla \cdot V = 0, \]

where \(\rho\) is the density of the fluid assumed to be constant.
Assuming \((u, v, w)\) as the velocity components along the cylindrical system of axes, the boundary conditions of the problem are:

**Case I:**

\[
\begin{align*}
    u &= 0, \quad v = 0, \quad w = 0 \quad \text{at} \quad z = 0, \\
    u &= r \Omega, \quad v = r \Omega, \quad w = 0 \quad \text{at} \quad z = z_0.
\end{align*}
\]  

(6)

**Case II:**

\[
\begin{align*}
    u &= 0, \quad v = r \Omega, \quad w = 0 \quad \text{at} \quad z = 0, \\
    u &= ar, \quad v = 0, \quad w = 0 \quad \text{at} \quad z = z_0.
\end{align*}
\]  

(7)

Following Sharma and Gupta [3] and using (3), we take the following non-dimensional form of the velocity components satisfying the continuity equation (5) and that of pressure as

\[
\begin{align*}
    u &= r \Omega F' (\zeta), \\
    v &= r \Omega G (\zeta), \\
    w &= -2z_0 \Omega F (\zeta),
\end{align*}
\]  

and

\[
    P = \Omega \mu_i \left[-P_i (\zeta) + \frac{r^2}{z_0} \left[ R \zeta (F \zeta^2 + G \zeta^2) + \lambda \right] \right],
\]  

(8)

where \(F(\zeta)\) and \(G(\zeta)\) are non-dimensional functions of the dimensionless variable \(\zeta = \frac{z}{z_0}\) and \(R = \left(\frac{\Omega z_0^2}{\nu_i}\right)\) is Reynolds number. The dimensionless quantity \(T = \frac{V}{z_0^2}\) is the ratio of the elastico-viscous and the inertial effects and may be termed as elastico-viscous parameter, \(\lambda\) is a constant and primes denote differentiation with respect to \(\zeta\).

Substituting (8) and (9) in the equation of motion (4) and using equation (1)-(3) we obtain:

\[
\begin{align*}
    R \left( F \zeta^2 - G \zeta^2 - 2FF'' \right) &= F \zeta^2 - 2TR \left[ F \zeta^2 + G \zeta^2 + 2FF' - 2F'F'' \right] - 2\lambda, \\
    2R \left( F'G - FG' \right) &= G \zeta^2 - 2TR \left[ (FG'' - F'G') \right], \\
    4R \left( FF' \right) &= P_i - 2F'' + 4TR \left[ 4F'F'' + FF'' \right].
\end{align*}
\]  

(10)

The boundary conditions now take the form:

**Case I:**

\[
\begin{align*}
    F &= 0, \quad F' = 0, \quad G = 0, \quad \text{at} \quad \zeta = 0, \\
    F &= 0, \quad F' = M, \quad G = 1, \quad \text{at} \quad \zeta = 1.
\end{align*}
\]  

(13)

**Case II:**

\[
\begin{align*}
    F &= 0, \quad F' = 0, \quad G = 1, \quad \text{at} \quad \zeta = 0, \\
    F &= 0, \quad F' = M, \quad G = 0, \quad \text{at} \quad \zeta = 1.
\end{align*}
\]  

(14)

Where \(M = \frac{a}{\Omega}\) is a dimensionless forced parameter assumed to be small \((M \leq 1)\).

Since \(\lambda\) is a constant, we differentiate equation (10) to eliminate \(\lambda\) to find

\[
F'''' = 2R \left[ F' (G'G'' + 2FF'') - (GG'' + FF'') \right],
\]  

(15)
The unknowns involved in the velocity field are to be determined from the equations (15) and (16) satisfying the boundary conditions (13) and (14).

3. Series Solution

For small values of Reynolds number \( R_z \), a regular perturbation technique can be developed by expending \( F, G \) and \( \lambda \) in ascending powers of \( R_z \) for both the cases as follows:

\[
F = \sum_{n=0}^{\infty} f_n R_z^n, \quad G = \sum_{n=0}^{\infty} g_n R_z^n \quad \text{and} \quad \lambda = \sum_{n=0}^{\infty} \lambda_n R_z^n.
\]  

(17)

4. Numerical Solution

The numerical solution of \( F' \) and \( G \), the non-dimensional radial and transverse components of velocity have also been obtained by the finite difference method for \( R_z = 0.5, 5 \) and 10. The radial and transverse components have been graphically represented through figs. 1 and 2 for different values of \( R_z \), \( T \), and \( M \).

5. Results and Discussions

The numerical solution of \( F' \) and \( G \), the non-dimensional radial and transverse components of velocity have been obtained by the finite difference method for \( R_z = 0.5, 5 \) and 10. The radial and transverse components have been graphically represented through figs. 1 and 2 for different values of \( R_z \), \( T \), and \( M \).

In the first case, the variation of the radial component of velocity \( U(\zeta) \) as a function of \( \zeta \) for different values of the visco-elastic parameter \( T \), Reynolds number \( R_z \) and forced parameter \( M \) have been calculated and shown through fig. 1. It is found that for fixed value of \( M=1 \), the radial velocity component decreases with increase in visco-elastic parameter \( T \) near the lower disc and increases near the upper disc for the values of \( T= 0, 1 \) and \( R_z =5,10 \). Similarly for fixed value of \( T=1 \) and increasing values of \( M \), the effects are obviously opposite.

In the second case, the variation of the transverse component of velocity \( V(\zeta) \) for different values of the visco-elastic parameter \( T \), Reynolds number \( R_z \) and forced parameter \( M \) have been calculated and shown through fig. 2. It is found that the transverse component of velocity increases near the lower disc and decreases near the upper disc with fixed value of \( M \) at \( M=1, T=1 \), 2 and \( R_z=0.5, 5 \) and 10.

6. Conclusion

In this paper, we have discussed the problem of forced flow of a visco-elastic second-grade fluid between two infinite discs, the system coinciding with the planes \( z=0 \) and \( z=z_0 \) respectively. The space between the discs is occupied by a visco-elastic second-grade fluid. Two cases have been considered; in the first case, the lower disc is stationary while the upper disc is rotating with constant angular velocity \( \Omega \) about the \( z \)-axis and also creating a symmetrical radial velocity \( ar \). In the second case, the lower disc rotates with the same constant angular velocity \( \Omega \) while the upper disc is stationary and creates a symmetrical
radial velocity $a r$. The constitutive equation, equation of motion and equation of continuity of the fluid together give rise to the set of highly non-linear system of equations which are solved by finite difference method and Newton-Raphson method. The variation of dimensionless radial and transverse component of velocity for two different cases viz.:

(i) when upper disc rotates and the lower disc is stationary and 
(ii) when lower disc rotates and upper disc is stationary

for different visco-elastic and forced parameters have been obtained & shown graphically through figs. 1 and 2. The effects of forced parameter $M$ and visco-elastic parameter $T$ have been discussed in detail. For fixed value of $M=1$, the radial velocity component is found to decrease with an increase in visco-elastic parameter $T$ near the lower disc and increase near the upper disc for the values of $T=0, 1$ and $R_z=5, 10$. Similarly for fixed value of $T=1$ and increasing values of $M$, the effects are obviously opposite. In second case, the variation of the transverse component of velocity $V(\zeta)$ for different values of the visco-elastic parameter $T$, Reynolds number $R$, and forced parameter $M$ have been calculated and shown through fig. 2.

It is found that the transverse component of velocity increases near the lower disc and decreases near the upper disc with fixed value of $M$ at $M=1, T=1, 2$ and $R_z=0.5, 5$ and 10. For $T=0$, corresponding expressions for viscous case can be obtained. For $M=0$, we get the results for flow of second-grade fluid when one disc rotates and the other is at rest. The transverse shearing stress on the lower disc, the radial pressure variation on the lower disc between any radii $\xi_1$ and $\xi_2$, the average normal force on the lower disc up to radius $\xi_0$ have also been obtained.

References

