Mathematical Modelling of Flow in Narrowing Systems

A.D. Patel$^1$, I.A. Salehbhai$^1$, S.K. Meher$^1$, A.K. Shukla$^1$, M.N. Mehta$^1$, V.K. Katiyar$^2$.

$^1$Department of Mathematics, S.V. National Institute of Technology, Surat, Gujarat, India.
$^2$Department of Mathematics, IIT, Roorkee, UA, India.

Abstract

Narrowing of pipeline network is an important aspect in drinking water distribution systems, Sewage system and in oil-well techniques and also blood flow through artillery system. In the proposed problem, a flow equation in simple pipeline network has been studied to solve the velocity flow, flux and shear stress on the wall. The deposition causing narrowing has been replaced by using sinusoidal model with axial velocity by applying fractional calculus techniques yields the results in terms of Bessel’s function under the sign of summation.

1. Introduction

The continuity and Navier-Stokes equations [7] for incompressible flow are:

\[
\mathbf{\nabla} \cdot \mathbf{V} = 0
\]

\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} \right) = -\mathbf{\nabla} p + \mu \mathbf{\nabla}^2 \mathbf{V}
\]

where $D$ is substantial derivative, $V$ is velocity vector, $t$ is time, $\rho$ is density, $p$ is pressure, $\mu$ is kinematic viscosity.

Expanded form of equation (1.2) for $r, \theta$ and $z$ directions can be written as

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial \theta} + w \frac{\partial u}{\partial z} = -\left( \frac{1}{\rho} \frac{\partial p}{\partial r} + \mu \left( \frac{1}{\rho} \frac{\partial \left( ru \right)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} - \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial \theta} + w \frac{\partial v}{\partial z} = -\left( \frac{1}{\rho} \frac{\partial p}{\partial r} + \mu \left( \frac{1}{\rho} \frac{\partial \left( rv \right)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2} + \frac{2}{r^2} \frac{\partial u}{\partial \theta} \right)
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + v \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} = -\left( \frac{1}{\rho} \frac{\partial p}{\partial z} + \mu \left( \frac{1}{\rho} \frac{\partial \left( rw \right)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2} \right)
\]

El-Shahed et al [3, 4] and Shukla et al [12] used the fractional calculus approach in fluid dynamics. The governing equations have been described by fractional partial differential equation and the exact solution of these equations have been obtained by using the discrete Laplace transform, Fourier transform and some well-known Special functions. Heymans Nicole and Podlubny Igor [5], Podlubny Igor [8] and [9] have discussed the physical interpretation of the Riemann-Liouville fractional differentiation and integration and proposed it in terms of inhomogeneous and changing (non-static, dynamic) time scale.
2. Mathematical formulation of the problem and solution

Let a long circular cylinder in which fluid is at rest initially, suddenly a constant pressure gradient is imposed along the axis of the cylinder, due to the pressure gradient fluid is set into the motion (constant $\rho$ and $\mu$). Let $Z$ as the direction of the axis of cylinder along which the flow takes place and let $r$ be the radial direction outward from the $Z$-axis and consider, the flow is fully developed and axially symmetric. We also assume that there are some depositions of thickness $\delta$ on the wall of the cylinder which causes the narrowing the system, which satisfies the equation of the thickness due to deposition as

$$R = R_0 - \frac{\delta}{2} \left( 1 + \cos \frac{\pi z}{z_0} \right),$$

where $\delta$ is the deposition thickness, $R_0$ is the distance from axis of the cylindrical boundary and $z$ is the distance from $z = 0$ to the point of calculation $P$.

If $z = 0$, $R = R_0 - \delta$ (i.e. at the centre of the deposition) and if $z = z_0$, $R = R_0$ (i.e. at the starting and ending point of the deposition).

![Fig. 1: Schematic diagram of narrowing system](image)

Here we consider the $Z$-momentum equation with respect to inhomogeneous time as;

$$\rho \frac{\partial^\alpha w}{\partial t^\alpha} = - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right), 0 < \alpha < 1.$$

If $\alpha \to 1$, then above equation reduces to simplified momentum equation where $\mu$ is kinematic viscosity.

Initial condition and boundary conditions are considered as:

$$w(r,0) = 0$$

$$w(r,t) = 0 \text{ at } r = R$$

$$w(0,t) \text{ is finite at } r = 0$$
The method integral transform is used to obtain the solution of the problem, which gives the velocity of the fluid as follows:

\[ w(r,t) = \frac{2Pt^\alpha}{\rho R} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n^\alpha J_1(\lambda_n R)} E_{\alpha,\alpha+1}(-\nu \lambda_n^2 t^\alpha) \]

If \( \alpha \rightarrow 1 \) then above equation reduces to

\[ w(r,t) = \frac{2Pt}{\rho R} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n^2 J_1(\lambda_n R)} E_{1,2}(-\nu \lambda_n^2 t) \]

3. Conclusion

The concept of fractional calculus approach is extended to the study of narrowing system problem. The exact solution of the fractional partial differential equation is obtained in term of well-known Bessel function by using Laplace transform, Hankel transform and Special functions with appropriate initial and boundary conditions. The solution is obtained in the form of Mittag-Leffler function. We can also obtain Shear Stress and Flux in inhomogeneous time scale. This method may be more useful than conventional methods for further theoretical analysis and applications of complex forms of Navier-Stokes and energy equations.

Reference