Mathematical Approach to the Human-Brain Functions

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A cell body initiates action potentials in axons at a frequency or ‘firing rate’ which depends in turn on the ion leaks of the cell, which in turn depends on the chemical signal from other cells. Thus a particular cell may crudely be characterized by an ‘output’ firing rate which is caused by the sum of all the inputs to a cell. The typical experimental input–output may be sigmoid, with firing rates less than zero or greater than a maximum of about 500-sec⁻¹ not possible. This description suppresses all the details of the time structure of the firing pattern. We will presume that the dominant information in the system is contained in an appropriate short-term mean of the firing rate. There are circumstances in which the quantal nature of the spikes and their time separations will be essential. But just as the classical electromagnetic theory, which neglects the quantal properties of light, is awfully useful in an appropriate realm, (such as understanding how Television works) we believe the firing rate description will also have a useful domain.

1. Introduction

The brain is a large physical system, and the principles of its operations must ultimately be describable in physical terms. There are enormous obstacles to accomplishing this. Unlike the physicist’s large systems, which normally consist of huge number of essentially identical simple objects, the brain is very complex. Biological evolutions have endowed a brain with tens or even hundreds of different types of cells. The brain has a rich structure of regions, and cellular connection architecture within each region. If understanding the functioning of a brain entails for mammalian brains of different individuals are far from identical as well as extremely large.

1.1 Possible properties of neurons

The resting potential inside a typical axon or nerve process is about -100 milli volts with respect to the outside. Such axons are well known to be capable of solitary waves or ‘spikes’ of electrochemical activity propagating along an axon. These spikes are actively propagated without loss, the energy coming the electrochemical gradients across the cell membrane. The input to a given neuron comes from the other neurons, which make synapses with it. We will write

\[ \text{Input to } i = \sum_j T_{ij} V_j \]

(1)

where \( V_j \) is the firing rate of the neuron \( j \) and \( T_{ij} \) is the strength of synaptic connection from neuron \( j \) to neuron \( i \). In real practice, both signs of synapses are possible, but each neuron \( j \) tends to specialize in one sign of connection. At present, this is a needless
complication, and our model neurons will each be allowed to have $T_{ij}$ of either sign for all $i$ and for all $j$. To describe a flow in state space, a time dynamics of the neural states must be given. We use a delayed, stochastic, dynamics in which each neuron $i$ stochastically, but at mean rate we examine the sum of its inputs. If at that instant the sum of the inputs

$$\text{Input to } i = I_i = \sum_j T_{ij} V_j$$

(2)

The algorithm seems very arbitrary. Its choice was dictated by a desire to include noise, asynchronous operation, and propagation delays effects, while at the same time keep the mathematics and simulation problem tractable. The algorithm can be argued to represent qualitatively such effects without the need to include irrelevant details. Some assumptions must be made about the matrix $T_{ij}$ in order to see whether the algorithm yields a flow with stable limit points. In a brain or even a simple insect ganglion, typical cell are connected to about 1,000 to 10,000 other nearby cells. While there is also some long-range connectivity, the dominant connectivity is to near by cells. One might take a collection of about 1,000 cells, each capable of getting signals from each (or many) of the others as a plausible fundamental processing unit. The short-time behavior of such a finite and densely interconnected unit should have relevance to brain function.

1.2 Results

Simulations were carried out on a digital computer for $N=10$ and for $N=30$ neurons, with varying numbers of neurons connected to others. Ten turns out to be too small a number to demonstrate collective behavior well. For $N=30$, a randomly chosen would produce a flow with about there dominating limit ‘points’. Sometimes the limiting states were not single states, but instead small cycles like ABAB--. Some of the limit ‘points’ turned out to be small limit regions in which the state wandered stochastically over a small region of state space. If the closeness of two digital states, each being a word of $N$ bits, is defined as the hamming distance between them (the number of digits in which they differ), all the states are seen in the limit region were within about 4 units of each other. (Two random state would be about 15 units apart for the case $N=30$). The dominating limit points were the end points of typical runs from random initial states. There are, in addition, other limit points, which are very unlikely to be reached from random starting states.

The change in $H$ due to changing $V$ while keeping all Thus, the dynamics defined on the state space of $N$-bit words by a random and the stochastic algorithm produces the kind of flow in state space which is essential to content-addressable memory. There are no problems like flows which wandered chaotically through space in an ergodic fashion.

1.3 Necessary of limit point

Consider the function

$$H = -\sum_{i,j} T_{ij} V_i V_j$$

(3)

$$\Delta H_{io} = -\left(\sum_j T_{ijo} V_j \right) V_{io} - \left(\sum_j T_{iio} V_j \right) \Delta V_{io}$$

(4)
According to the algorithm, $\Delta V_{io}$ can be positive only when $T_{ioj} V_j > 0$, and can be negative only when this sum is less than zero. Thus, following the algorithm, $\Delta H_{io}^1$ is always negative. (It is assumed throughout that there is no self-coupling, i.e. $T_{ii} = 0$).

For the special case of a symmetric synapse matrix $T_{ij} = T_{ji}$, one immediately notes that $\Delta H_{io}^1 = \Delta H_{io}^2$.

Thus, for a symmetric matrix the algorithm is minimizing a bounded function, and must converge on a minimum which is at least locally stable. When $T_{ij}$ is not symmetric, no such constraint exists. $\Delta H_{io}^2$ is a more or less random change at each step while $\Delta H_{io}^1$ is monotonic. Such a mode of behavior would be present for a symmetric system which was also in contact with a thermal reservoir, so that a ‘free energy’ function is what was truly being minimized, and the energy tends to a compromise between lowering itself as much as possible and the phase space available (at finite temperature) for low energy excitations. Simulations show that when $T$ is not symmetric, a system started in a random state (with $N$ near zero) rapidly ‘cools’ attaining a large negative $H$, then shows fluctuations in $H$ around that value while wandering in a small final region.

### 1.4 Inserting chosen memories

In the study of linear associative memory the problem of constructing or learning a matrix which will generate a given set of vector input - output relations has been extensively dealt with. An understanding of this study suggests how to build in our problem in order that specific vector states $\{ {V_i}^s \}_{i=1}^N$ (naturally $V_i^0 = 1$ or 0 for all $i$ and $s$) be the stable memories. The following explanation is applicable for the special case $I_i = 0$, and for state with, on the average $V_i^8 = 1/2$. Obvious simple alterations are necessary when this average is smaller.

The biological idea behind this modeling is the modifiable synapse at the basis of memory.

In addition, we will pick the memories as random vectors, thus random N-bits digital words. (This supposition of randomness is appropriate if the information being stored has been optimally coded. Thus, for example, coding DNA sequences appear roughly random. It could also be taken as representing out lack of knowledge of how memories are stored), we consider

$$T_{ij} = 4 \sum_s (V_i^s - 1/2)(V_j^s - 1/2)$$

The stability of a particular state $s$ can be treated by examining

$$\sum_j T_{ij} V_i^s = 1/4 \sum_i \sum_{s=3}^8 (2V_i^s - 1)(V_j^s - 1)V_{ji}^s + (2V_i^s - 1)\sum_j (2V_j^s - 1)V_j^s$$

whose sign determines whether $V$ should be 0 or 1. The first term, the double sum has an average value of zero, and its value has no correlation with. In the second term, the sum has an expectation value of $N/2$. Thus on the average, the net second term is $N/2$ if is +1, and $-N/2$ if $= 0$. Second terms precisely stabilizes the state $S$. 

$$= \Delta H_{io}^1 + \Delta H_{io}^2$$

(5)
The incremental rule
\[ \Delta T_{ij} = (2V_i^s - 1)(2V_j^s - 1) \tag{8} \]
when a new memory \( V^s \) is to be inserted is very attractive, since it does not depend on what memories has been previously store. In addition synapse \( T_{ij} \) needs only information on \( V_i^s \) and \( V_j^s \) which is locally available to it and needs no information about the global nature of a memory. A modification of this rule is essential to forget old memories in order to incorporate new ones. Many such modifications are possible, but a particularly simple one is

\[ \Delta T_{ij} = (2V_i^s - 1)(2V_j^s - 1) \tag{9} \]
if such an increment leaves \( |T_{ij}| < T_{\text{max}} \),
if not, \( \Delta T_{ij} = 0 \)

If \( T_{\text{max}} \approx 4 \), then new memories are written into memory with almost the usual efficacy, while old memories become unstable and disappear. Let the robustness of the memory storage to change is \( T_{ij} \) when a matrix \( T_{ij} \) is constructed in the usual way, the new matrix \( T_{ij}^{\text{new}} = \text{sign}(T_{ij}) \),

\[ \sum_j T_{ij} V_j, \tag{10} \]
has the same memory state as \( T \), with only a slight increase in noise or errors. This fact allows the use of a matrix consisting only of two symbols 1 and -1, or by appropriate transformations, 0 and 1. The major operation in calculating \( I_i \) is the multiplications and additions in

\[ \sum_j T_{ij} V_j, \tag{11} \]

When both \( T \) and \( V \) consist only of 0 or 1’s this can be done with logical and no multiplication, a small, and a few additions.

### 1.5 The sense of time and time-order

For a random matrix, a sense of the flow of time is generated by the tendency for \( H \) to decrease in time. The essential change in \( T_{ij} \) to recall a sequence \( V^1, V^2, \ldots \) in order is to add a term

\[ \Delta_s (2V_i^{s+1} - 1)(2V_j^s - 1), \tag{12} \]
to \( T_{ij} \). This term has the same kind of locality at site \( ij \) as the usual one, and is a trivial generalization. If before we were to think of \( T_{ij} \) as a time average,

\[ T_{ij} = 4 \int dt(V_i(t) - 1/2)(V_j(t) - 1/2), \tag{13} \]
the additional term is a time delay, replacing \( V_j(t) \) by \( V_j(t - T) \) while leaving \( V_j(t) \) unchanged. Any biochemical hardware conceptually capable of producing the previous \( T_{ij} \) could equally easily be induced to generate the new prescription.
1.6 Prospects
The work prospects on some things which need to be done, on speculations on what the system might do. The replacement of the sigmoid firing rate versus input by a step seemed a diastase oversimplification. The structured of the mathematics is such as to allow the analyses near points of stability when the true sigmoid shape is used. This has not been carried out in detail, but what has been done shows that the step can be replaced by a smooth sigmoid with no sudden change in stability.

Single neurons have been allowed to make both excitatory and inhibitory synapses. If there are two classes of cells, each of which makes only one type of synapse, it alters the signal to noise a little but does not change the basic behavior. A major biological reason for hoping that collective aspects will be important processing tools in higher nervous function concerns the evolution of intelligence. If it is necessary to evolve a computer-brain connection by connection and to have the connections all completely organized like a Cray in order to function, it is hard to understand how intelligence evolved. If on the other hand large assemblies of cells have important computational properties when their cellular properties are correctly chosen, then the evolutionary problem is much simpler.

There is considerable modeling in neurobiology, which is less directly related to issues raised here, but could be applied in better ways, which breaks through the study of this kind.

References